Some properties of row-adjusted meet and join matrices

7.9.2011

Mika Mattila* and Pentti Haukkanen
School of Information Sciences
FI-33014 University of Tampere, Finland

Abstract Let $(P, \preceq)$ be a lattice, S a finite subset of P and $f_1, f_2, \ldots, f_n$ complex-valued functions on P. We define row-adjusted meet and join matrices on S by $(S)_{f_1,\ldots,f_n} = (f_i(x_i \wedge x_j))$ and $[S]_{f_1,\ldots,f_n} = (f_i(x_i \vee x_j))$. In this paper we determine the structure of the matrix $(S)_{f_1,\ldots,f_n}$ in general case and in the case when the set S is meet closed we give bounds for $\mathrm{rank}(S)_{f_1,\ldots,f_n}$ and present expressions for $\det(S)_{f_1,\ldots,f_n}$ and $(S)^{-1}_{f_1,\ldots,f_n}$. The same is carried out dually for row-adjusted join matrix of a join closed set S.

Key words and phrases: Meet matrix, Join matrix, GCD matrix, LCM matrix, Smith determinant
AMS Subject Classification: 11C20, 15B36, 06B99

*Corresponding author. *Tel.:* +358 31 3551 7581, *fax:* +358 31 3551 6157
E-mail addresses: mika.mattila@uta.fi (M. Mattila),
pentti.haukkanen@uta.fi (P. Haukkanen)

1 Introduction

In 1876 Smith [16] presented a formula for the determinant of the $n \times n$ matrix $((i, j))$, having the greatest common divisor of i and j as its ij element. During the 20th century many other results concerning matrices with similar structure were published, see for example [7, 12, 19]. In 1989 Beslin and Ligh [4] introduced the concept of a GCD matrix on a set S, where $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^+$ with $x_1 < x_2 < \cdots < x_n$ and the GCD matrix (S) has (x_i, x_j) as its ij entry. Since then numerous publications have appeared in order to universalize the concept of GCD matrix. For example, Haukkanen [5] and Luque [13] consider the determinants of multidimensional generalizations of GCD matrices and Hong, Zhou and Zhao [8] study power GCD matrices for a unique factorization domain.

Poset theoretic generalizations of GCD matrices were first introduced by Lindström [11] and Wilf [18]. In these generalizations $(P, \preceq)$ is a poset, f is a function $P \to \mathbb{C}$, $S = \{x_1, x_2, \ldots, x_n\} \subset P$, $x_i \preceq x_j \Rightarrow i \leq j$ and $(S)_f$ is an $n \times n$ matrix with $f(x_i \wedge x_j)$ as its ij element. These matrices are referred to as meet matrices. The papers by Lindström [11] and Wilf [18] arose from needs for combinatorics and became possible since Rota [15] had previously developed his famous theory on Möbius functions. Rajarama Bhat [14] and Haukkanen [6] were the first to investigate meet matrices systematically, presenting many important properties of ordinary GCD matrices in terms of meet matrices. In [10] Korkee and Haukkanen define and study the join matrix $[S]_f$ of the set S with respect to f, where $f(x_i \vee x_j)$ is the ij element of the matrix $[S]_f$.

During the last ten years the concept of meet matrix has been generalized even further in many different ways. Korkee [9] studies the properties of a matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$, which yields both the matrix $(S)_f$ and $[S]_f$ as its special case. A totally different approach is taken by Altinisik, Tuglu and Haukkanen in [2], when they define meet and join matrices on two subsets X and Y of P. A further idea of generalization is presented by Bege [3] as he studies yet another GCD related matrix $(F(i, (i, j)))$, where $F(m, n)$ is an arithmetical function of two variables. For present purposes it is convenient to use a slightly different notation. For every $i \in \mathbb{Z}^+$ we define an arithmetical function f_i of one variable by

$$f_i(m) = F(i, m) \quad \text{for all } m \in \mathbb{Z}^+. \tag{1.1}$$

With this notation Bege's matrix takes the form

$$\begin{bmatrix} f_1((1,1)) & f_1((1,2)) & \cdots & f_1((1,n)) \\ f_2((2,1)) & f_2((2,2)) & \cdots & f_2((2,n)) \\ \vdots & \vdots & \ddots & \vdots \\ f_n((n,1)) & f_n((n,2)) & \cdots & f_n((n,n)) \end{bmatrix}. \tag{1.2}$$

In order to distinguish between this and the numerous other generalizations

of GCD matrices, this matrix is referred to as the *row-adjusted GCD matrix of the set* $\{1, 2, \ldots, n\}$. This notation also enables us to define row-adjusted meet and join matrices.

Definition 1.1. Let $(P, \preceq)$ be a lattice, $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of P with $x_i \preceq x_j \Rightarrow i \leq j$ and $f_1, f_2, \ldots, f_n$ be complex-valued functions on P. The row-adjusted meet matrix of the set S is the $n \times n$ matrix $(S)_{f_1,\ldots,f_n}$, which has $(f_i(x_i \wedge x_j))$ as its ij element. Similarly, the row-adjusted join matrix $[S]_{f_1,\ldots,f_n}$ has $(f_i(x_i \vee x_j))$ as its ij element.

More explicitly,

$$(S)_{f_1,\ldots,f_n} = \begin{bmatrix} f_1(x_1 \wedge x_1) & f_1(x_1 \wedge x_2) & \cdots & f_1(x_1 \wedge x_n) \\ f_2(x_2 \wedge x_1) & f_2(x_2 \wedge x_2) & \cdots & f_2(x_2 \wedge x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_n \wedge x_1) & f_n(x_n \wedge x_2) & \cdots & f_n(x_n \wedge x_n) \end{bmatrix} \tag{1.3}$$

and

$$[S]_{f_1,\ldots,f_n} = \begin{bmatrix} f_1(x_1 \vee x_1) & f_1(x_1 \vee x_2) & \cdots & f_1(x_1 \vee x_n) \\ f_2(x_2 \vee x_1) & f_2(x_2 \vee x_2) & \cdots & f_2(x_2 \vee x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_n \vee x_1) & f_n(x_n \vee x_2) & \cdots & f_n(x_n \vee x_n) \end{bmatrix}. \tag{1.4}$$

It turns out that there are some results concerning the matrix $(S)_{f_1,\ldots,f_n}$ to be found in the literature by Lindström [11] and Luque [13]. When the notation is the same as defined in (1.1), these results can easily be applied to Bege's matrix.

Unlike the ordinary meet and join matrices, the matrices $(S)_{f_1,\ldots,f_n}$ and $[S]_{f_1,\ldots,f_n}$ are usually not symmetric. There are also many other key properties of meet and join matrices that do not hold for row-adjusted meet and join matrices. Hence, neither the traditional methods of meet and join matrices works in the study of these row-adjusted matrices.

Remark 1.1. In the case when $f_1 = f_2 = \cdots = f_n = f$, we have $(S)_{f_1,\ldots,f_n} = (S)_f$ and $[S]_{f_1,\ldots,f_n} = [S]_f$.

Remark 1.2. Taking the transpose of a row-adjusted meet or join matrix results in a *column-adjusted* meet or join matrix. Therefore the results concerning row-adjusted meet and join matrices can easily be translated for column-adjusted meet and join matrices using this connection.

At the end of his paper Bege [3] presents an open problem regarding the structure and the determinant of the matrix $(F(i, (i, j)))$. It appears that the question about the determinant could be solved using Lindström's result in [11]. In this paper we present a more systematic investigation of

the structure of $(S)_{f_1,\ldots,f_n}$ and $[S]_{f_1,\ldots,f_n}$ in general case. Then by using this knowledge we are able to find a different proof for Lindström's determinant formula and also prove some other results concerning the rank and inverse of these matrices.

2 Preliminaries

Let $(P,\preceq)$ be a lattice, $S=\{x_1,x_2,\ldots,x_n\}$ a finite subset of P and

$$f_1,f_2,\ldots,f_n : P\to\mathbb{C}$$

complex-valued functions on P. We also assume that the elements of S are distinct and arranged so that

$$x_i\preceq x_j \Rightarrow i\leq j.$$

The set S is said to be *meet closed* if $x\wedge y\in S$ for all $x,y\in S$. In other words, the structure $(S,\preceq)$ is a meet semilattice. The concept of *join closed set* is defined dually.

Let $D=\{d_1,d_2,\ldots,d_m\}$ be another subset of P containing all the elements $x_i\wedge x_j$, $i,j=1,2,\ldots,n$, and having its elements arranged so that

$$d_i\preceq d_j \Rightarrow i\leq j.$$

Now for every $i=1,2,\ldots,n$ we define the function Ψ_{D,f_i} on D inductively as

$$\Psi_{D,f_i}(d_k)=f_i(d_k)-\sum_{d_v\prec d_k}\Psi_{D,f_i}(d_v), \tag{2.1}$$

or equivalently

$$f_i(d_k)=\sum_{d_v\preceq d_k}\Psi_{D,f_i}(d_v). \tag{2.2}$$

Thus we have

$$\Psi_{D,f_i}(d_k)=\sum_{d_v\preceq d_k}f_i(d_v)\mu_D(d_v,d_k), \tag{2.3}$$

where μ_D is the Möbius function of the poset $(D,\preceq)$, see [1, Section IV.1] and [17, 3.7.1 Proposition.].

Let E_D be the $n\times m$ matrix defined as

$$(e_D)_{ij}=\begin{cases}1 & \text{if } d_j\preceq x_i,\\ 0 & \text{otherwise.}\end{cases} \tag{2.4}$$

The matrix E_D may be referred to as the incidence matrix of the set D with respect to the set S and the partial ordering $\preceq$.

Finally, we need another $n\times m$ matrix $\Upsilon=(\upsilon_{ij})$, where

$$\upsilon_{ij}=(e_D)_{ij}\Psi_{D,f_i}(d_j). \tag{2.5}$$

In other words, if Ξ is the $n\times m$ matrix having $\Psi_{D,f_i}(d_j)$ as its ij element, then $\Upsilon=E_D\circ\Xi$, the Hadamard product of the matrices E_D and Ξ.

3 A structure theorem

In this section we give a factorization of the matrix $(S)_{f_1,\ldots,f_n}$, which then enables us to derive formulas for the rank, the determinant and the inverse of the matrix $(S)_{f_1,\ldots,f_n}$.

Theorem 3.1. *We have*

$$(S)_{f_1,\ldots,f_n} = \Upsilon E_D^T = (E_D \circ \Xi) E_D^T. \tag{3.1}$$

Proof. By (2.2), (2.4) and (2.5) the ij element of $(S)_{f_1,\ldots,f_n}$ is

$$f(x_i \wedge x_j) = \sum_{d_v \preceq x_i \wedge x_j} \Psi_{D,f_i}(d_v) = \sum_{k=1}^{m} (e_D)_{ik} \Psi_{D,f_i} (e_D)_{jk}, \tag{3.2}$$

which is the ij element of the matrix ΥE_D^T. □

Remark 3.1. It is possible to define row-adjusted meet and join matrices $(X, Y)_{f_1,\ldots,f_n}$ and $[X, Y]_{f_1,\ldots,f_n}$ on two sets X and Y by $((X, Y)_{f_1,\ldots,f_n})_{ij} = f_i(x_i \wedge y_j)$ and $([X, Y]_{f_1,\ldots,f_n})_{ij} = f_i(x_i \vee y_j)$. It would be possible to generalize Theorem 3.1 for these matrices, but the methods used in the proofs of the other theorems do not work in this general case.

Remark 3.2. In the case when the set S is meet closed Theorem 3.1 also provides an effective way to calculate all the necessary values $\Psi_{S,f_i}(x_j)$ as follows. In this case $D = S$ and both E_S and Υ are square matrices of size $n \times n$. Since E_S is also invertible, from equation (3.1) we obtain

$$\Upsilon = (S)_{f_1,\ldots,f_n} (E_S^T)^{-1}, \tag{3.3}$$

which gives the values of $\Psi_{S,f_i}(x_j)$. Here the matrix E_S^T is the matrix associated with the zeta function ζ_S of the set S (see [1, p. 139]), and thus the matrix $(E_S^T)^{-1}$ is the matrix of the Möbius function of the set S and has $\mu_S(x_i, x_j)$ as its ij element.

The following example gives a solution for the first part of Bege's problem.

Example 3.1. The row-adjusted GCD matrix of the set $S = \{1, 2, \ldots, n\}$ is the product of the matrices $\Upsilon = (\upsilon_{ij})$ and E_S^T, where

$$(e_S)_{ij} = \begin{cases} 1 & \text{if } j \mid i, \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

and

$$\upsilon_{ij} = (e_S)_{ij} \Psi_{S,f_i}(j) = (e_S)_{ij} \sum_{k \mid j} f_i(k) \mu\left(\frac{j}{k}\right) = (e_S)_{ij} (f_i * \mu)(j), \tag{3.5}$$

where $*$ is the Dirichlet convolution and μ is the number-theoretic Möbius function. It should be noted that here the notation $F(i, k) = f_i(k)$ is not only convenient but also enables the use of the Dirichlet convolution.

4 Rank estimations

In this section we derive bounds for $\operatorname{rank}(S)_{f_1,\ldots,f_n}$ in the case when the set S is meet closed. The rank of meet and join matrices or even GCD and LCM matrices has not been studied earlier in the literature.

Theorem 4.1. *Let S be a meet closed set and let k be the number of indices i with $\Psi_{D,f_i}(x_i) = 0$. Then the following properties hold.*

1. $\operatorname{rank}(S)_{f_1,\ldots,f_n} = 0$ *iff* $f_i(x_i \wedge x_j) = 0$ *for all* $i, j = 1, \ldots, n$.
2. *If* $k = 0$, *then* $\operatorname{rank}(S)_{f_1,\ldots,f_n} = n$.
3. *If* $k > 0$, *then*
$$n - k \leq \operatorname{rank}(S)_{f_1,\ldots,f_n} \leq n - 1. \tag{4.1}$$

Proof. 1. Follows trivially.

2. By Theorem 3.1 we have
$$\operatorname{rank}(S)_{f_1,\ldots,f_n} = \operatorname{rank}\left(\Upsilon E_S^T\right). \tag{4.2}$$
Since in this case the matrices Υ and E_S are both triangular square matrices with full rank, the claim follows immediately.

3. Since multiplying with the invertible matrix E_S^T does not change the rank, we have
$$\operatorname{rank}(S)_{f_1,\ldots,f_n} = \operatorname{rank}\Upsilon. \tag{4.3}$$
To obtain the latter inequality we only need to note that since at least one of the diagonal elements of Υ equals zero, the rows of Υ cannot be linearly independent and thereby Υ cannot have a full rank. On the other hand, the $n-k$ rows with nonzero diagonal elements constitute a linearly independent set, from which we obtain the first inequality.

$\square$

In the case when the set S is meet closed and $f_1 = \cdots = f_n = f$ (that is in the case of ordinary meet matrix) the question of the rank becomes trivial. Namely, the matrix $(S)_f$ can be written as
$$(S)_f = E_S \Lambda E_S^T, \tag{4.4}$$
where $\Lambda = \operatorname{diag}(\Psi_{S,f}(x_1), \Psi_{S,f}(x_2), \ldots, \Psi_{S,f}(x_n))$, see [2, Theorem 3.1.]. Now by the same argument as in the proof of Theorem 4.1 we have
$$\operatorname{rank}(S)_f = \operatorname{rank}\Lambda = n - k. \tag{4.5}$$

The following two examples show that the bounds in Theorem 4.1 are the best possible under these assumptions. They also show that a large value of k may indicate a large decline of the rank of the row-adjusted meet matrix, but not necessarily.

Example 4.1. Let $x_1 = x_i \wedge x_j$ for all $i, j = 1, \ldots, n$, which implies that x_1 is the smallest element of S and the set $S \backslash \{x_1\}$ is an antichain. Now the set S is clearly meet closed, and for every $i = 2, \ldots, n$ we have

$$\Psi_{S,f_i}(x_i) = f_i(x_i) - f_i(x_1). \tag{4.6}$$

If $i > 1$ and we set $f_i(x_i) = f_i(x_1)$, then the ith column of Υ becomes the zero vector and thus for every $i > 1$ we may reduce the rank of the matrix $(S)_{f_1,\ldots,f_n}$ by one. Therefore if the first diagonal element of Υ is not zero, then rank $(S)_{f_1,\ldots,f_n} = n - k$.

Example 4.2. Let $(P, \preceq) = \mathcal{N}_5$ and $S = P$ as shown in Figure 1. Let

$$f_2(x_2) = f_3(x_1) = f_3(x_3) = f_4(x_3) = f_4(x_4) = f_5(x_4) = f_5(x_5) = 1 \tag{4.7}$$

and $f_i(x_j) = 0$ otherwise. Simple calculations show that $\Psi_{S,f_2}(x_2) = 1 \neq 0$,

$$\Psi_{S,f_1}(x_1) = \Psi_{S,f_3}(x_3) = \Psi_{S,f_4}(x_4) = \Psi_{S,f_5}(x_5) = 0, \tag{4.8}$$

and thereby $k = 4$. But on the other hand we have

$$(S)_{f_1,\ldots,f_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \tag{4.9}$$

and clearly rank $(S)_{f_1,\ldots,f_n} = 4$.

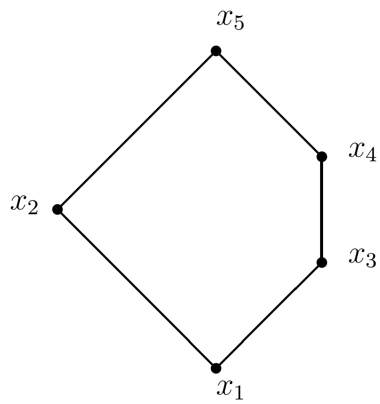

Figure 1: The lattice $\mathcal{N}_5$ and the choices of the elements of the set S.

5 Determinant formula

In this section we present a determinant formula for the matrix $(S)_{f_1,\ldots,f_n}$ when the set S is meet closed. This theorem is almost the same as that presented by Lindström [11]. It is possible to use the Cauchy-Binet equality to obtain a determinant formula for $(S)_{f_1,\ldots,f_n}$ in general case. Since it is similar to the case of usual meet matrix, we do not present it here.

Theorem 5.1 ([11], Theorem)**.** *If the set S is meet closed, then*

$$\det(S)_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi_{S,f_i}(x_i) = \prod_{i=1}^{n} \sum_{x_j \preceq x_i} f_i(x_j)\mu_S(x_j, x_i). \tag{5.1}$$

Proof. Since the set S is meet closed, we have $D = S$. Then the matrix E_S is a lower triangular square matrix having every main diagonal element equal to 1. The matrix Υ is a lower triangular square matrix with $\Psi_{S,f_1}(x_1), \Psi_{S,f_2}(x_2), \ldots, \Psi_{S,f_n}(x_n)$ as diagonal elements. Thus $\det E_S = 1$ and by Theorem 3.1 we have

$$\det(S)_{f_1,\ldots,f_n} = \det \Upsilon = \prod_{i=1}^{n} \Psi_{S,f_i}(x_i). \tag{5.2}$$

The second equality follows from (2.3).

$\square$

Remark 5.1. The original theorem by Lindström [11] is slightly more general since it does not require the assumption $x_i \preceq x_j \Rightarrow i \leq j$. As he states, the rows and columns of $(S)_{f_1,\ldots,f_n}$ can always be permuted in a way that does not change the determinant but makes the matrix $(S)_{f_1,\ldots,f_n}$ to fulfill this condition.

The following example gives a solution to the second part of Bege's problem.

Example 5.1. For the row-adjusted GCD matrix on the set $S = \{1, 2, \ldots, n\}$ we have

$$\det(\{1, 2, \ldots, n\})_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi_{S,f_i}(i) = \prod_{i=1}^{n} \sum_{j\,|\,i} f_i(j)\mu\left(\frac{i}{j}\right) = \prod_{i=1}^{n} (f_i * \mu)(i). \tag{5.3}$$

6 Inverse formula

In this section we study the inverse of the matrix $(S)_{f_1,\ldots,f_n}$ when the set S is meet closed. A formula for $(S)^{-1}_{f_1,\ldots,f_n}$ in general case could be obtained with the aid of meet matrices on two sets and the Cauchy-Binet equation. We do not, however, present the details here.

Theorem 6.1. *If the set S is meet closed, then the matrix $(S)_{f_1,\ldots,f_n}$ is invertible iff $\Psi_{S,f_i}(x_i) \neq 0$ for all $i = 1,\ldots,n$. Furthermore, in this case the inverse of $(S)_{f_1,\ldots,f_n}$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$b_{ij} = \sum_{k=j}^{n} \mu_S(x_i, x_k)\theta_{kj}, \tag{6.1}$$

where the numbers $\theta_{jj}, \theta_{j+1,j}, \ldots, \theta_{nj}$ are defined recursively as

$$\theta_{kj} = \begin{cases} \dfrac{1}{\Psi_{S,f_j}(x_j)} & \text{if } k = j, \\ -\dfrac{1}{\Psi_{S,f_k}(x_k)} \displaystyle\sum_{u=j}^{k-1} e_{ku}\Psi_{S,f_k}(x_u)\theta_{uj} & \text{if } k > j. \end{cases} \tag{6.2}$$

Proof. The first part follows directly from Theorem 5.1. To prove the second part we use Theorem 3.1 and we obtain

$$(S)^{-1}_{f_1,\ldots,f_n} = (E_S^T)^{-1}\Upsilon^{-1}. \tag{6.3}$$

In order to obtain the ij element of the matrix $(S)^{-1}_{f_1,\ldots,f_n}$ we only have to ascertain the ith row of $(E_S^T)^{-1}$ and the jth column of Υ^{-1}. As stated in Remark 3.2, the matrix $(E_S^T)^{-1}$ is the matrix associated with the Möbius function of the set S. Therefore its ith row is

$$\begin{bmatrix} 0 & \ldots & 0 & \underbrace{\mu_S(x_i, x_i)}_{=1} & \mu_S(x_i, x_{i+1}) & \ldots & \mu_S(x_i, x_n) \end{bmatrix}. \tag{6.4}$$

Now let $\Theta = (\theta_{ij})$ denote the inverse of Υ. By multiplying the jth row of Υ with the jth column of Θ, we obtain

$$\Psi_{S,f_j}(x_j)\theta_{jj} = 1. \tag{6.5}$$

Further, the multiplication of the kth row of Υ and the jth column of Θ results in

$$\sum_{u=j}^{k} e_{ku}\Psi_{S,f_k}(x_u)\theta_{uj} = 0. \tag{6.6}$$

Thus we obtain (6.2), and (6.1) follows when we multiply the matrices Θ and $(E_S^T)^{-1}$.

□

7 Formulas for row-adjusted join matrices

In this section the results presented in previous sections are translated for row-adjusted join matrices. The proofs of these dual theorems are omitted

for the sake of brevity. Row-adjusted join matrices (or even row-adjusted LCM matrices) have not previously been studied in the literature. As stated in Remark 1.2, the study of column-adjusted join matrices can easily be reverted to the study of row-adjusted join matrices via taking the transpose.

Let $D' = \{d'_1, d'_2, \ldots, d'_{m'}\}$ be a subset of P containing all the elements $x_i \vee x_j$, $i, j = 1, 2, \ldots, n$, and having its elements arranged so that

$$d'_i \preceq d'_j \Rightarrow i \leq j.$$

For every $i = 1, 2, \ldots, n$ we define the function Ψ'_{D',f_i} on D' inductively as

$$\Psi'_{D',f_i}(d'_k) = f_i(d'_k) - \sum_{d'_k \prec d'_v} \Psi'_{D',f_i}(d'_v), \tag{7.1}$$

or equivalently

$$f_i(d'_k) = \sum_{d'_k \preceq d'_v} \Psi'_{D',f_i}(d'_v). \tag{7.2}$$

Thus we have

$$\Psi'_{D',f_i}(d'_k) = \sum_{d'_k \preceq d'_v} f_i(d'_v)\mu_{D'}(d'_k, d'_v), \tag{7.3}$$

where $\mu_{D'}$ is the Möbius function of the poset $(D', \preceq)$, see [17, 3.7.2 Proposition.].

Let $E'_{D'}$ be the $n \times m'$ matrix defined as

$$(e'_{D'})_{ij} = \begin{cases} 1 & \text{if } x_i \preceq d'_j, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Finally, let $\Upsilon' = (\upsilon'_{ij})$ be the $n \times m'$ matrix, where

$$\upsilon'_{ij} = (e'_{D'})_{ij}\Psi'_{D',f_i}(d'_j). \tag{7.5}$$

Theorem 7.1.

$$[S]_{f_1,\ldots,f_n} = \Upsilon'(E'_{D'})^T. \tag{7.6}$$

Theorem 7.2. *Let S be a join closed set and let k be the number of indices i with $\Psi'_{D',f_i}(x_i) = 0$. Then the following properties hold.*

1. $\operatorname{rank}[S]_{f_1,\ldots,f_n} = 0$ *iff* $f_i(x_i \vee x_j) = 0$ *for all* $i, j = 1, \ldots, n$.

2. *If* $k = 0$, *then* $\operatorname{rank}[S]_{f_1,\ldots,f_n} = n$.

3. *If* $k > 0$, *then*

$$n - k \leq \operatorname{rank}[S]_{f_1,\ldots,f_n} \leq n - 1. \tag{7.7}$$

Theorem 7.3. *If the set S is join closed, then*

$$\det[S]_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi'_{S,f_i}(x_i) = \prod_{i=1}^{n} \sum_{x_i \preceq x_j} f_i(x_j)\mu_S(x_i, x_j). \tag{7.8}$$

Theorem 7.4. *If the set S is join closed, then the matrix $[S]_{f_1,\ldots,f_n}$ is invertible iff $\Psi'_{S,f_i}(x_i) \neq 0$ for all $i = 1, \ldots, n$. Furthermore, in this case the inverse of $[S]_{f_1,\ldots,f_n}$ is the $n \times n$ matrix $B' = (b'_{ij})$ with*

$$b'_{ij} = \sum_{k=1}^{j} \mu_S(x_k, x_i)\theta'_{kj}, \tag{7.9}$$

where the numbers $\theta'_{jj}, \theta'_{j-1,j}, \ldots, \theta'_{1j}$ are defined recursively as

$$\theta'_{kj} = \begin{cases} \dfrac{1}{\Psi'_{S,f_j}(x_j)} & \text{if } k = j, \\ -\dfrac{1}{\Psi'_{S,f_k}(x_k)} \displaystyle\sum_{u=k+1}^{j} e'_{ku}\Psi'_{S,f_k}(x_u)\theta'_{uj} & \text{if } j > k. \end{cases} \tag{7.10}$$

References

[1] M. Aigner, *Combinatorial Theory*, Springer–Verlag, New York, 1979.

[2] E. Altinisik, N. Tuglu and P. Haukkanen, Determinant and inverse of meet and join matrices, *Int. J. Math. Math. Sci. vol. 2007* (2007) Article ID 37580.

[3] A. Bege, Generalized GCD matrices, *Acta Univ. Sapientiae Math. 2, 2* (2010) 160-167.

[4] S. Beslin and S. Ligh, Greatest common divisor matrices, *Linear Algebra Appl. 118* (1989) 69-76.

[5] P. Haukkanen, Higher-dimensional GCD matrices, *Linear Algebra Appl. 170* (1992) 53-63.

[6] P. Haukkanen, On meet matrices on posets, *Linear Algebra Appl. 249* (1996) 111-123.

[7] P. Haukkanen, J. Wang and J. Sillanpää, On Smith's determinant, *Linear Algebra Appl. 258* (1997) 251-269.

[8] S. Hong, X. Zhou and J. Zhao, Power GCD matrices for a UFD. *Algebra Colloq. 16 no. 1* (2009) 71-78.

[9] I. Korkee, On a combination of meet and join matrices, *JP J. Algebra Number Theory Appl. 5(1)* (2005) 75-88.

[10] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions, *Linear Algebra Appl. 372* (2003) 127-153.

[11] B. Lindström, Determinants on semilattices, *Proc. Amer. Math. Soc. 20* (1969) 207-208.

[12] P. Lindqvist and K. Seip, Note on some greatest common divisor matrices, *Acta Arith. 84* (1998) 149-154.

[13] J.-G. Luque, Hyperdeterminants on semilattices, *Linear Multilinear Algebra Vol. 56, No. 3* (2008) 333-344.

[14] B. V. Rajarama Bhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl. 158* (1991) 77-97.

[15] G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2* (1964) 340-368.

[16] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc. 7* (1875-1876) 208-212.

[17] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Corrected reprint of the 1986 original, Cambridge studies in Advanced Mathematics, 49, Cambridge University Press, 1997.

[18] H. S. Wilf, Hadamard determinants, Möbius functions, and the chromatic number of a graph, *Bull. Amer. Math. Soc. 74* (1968) 960-964.

[19] A. Wintner, Diophantine approximations and Hilbert's space, *Amer. J. Math. 66* (1944) 564-578.